Geometry 5 - 3D Geometry Intro

TSS Math Club

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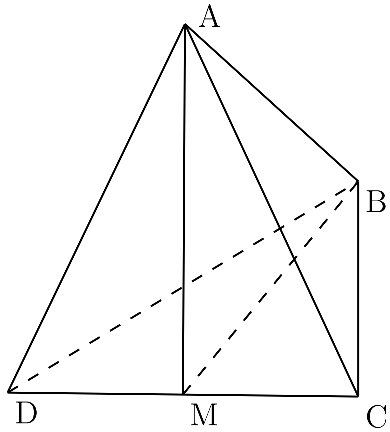
1 3D Geometry: Think 2D

1.1 Example

In a regular tetrahedron $ABCD$, M is the midpoint of CD. Find $\angle AMB$.

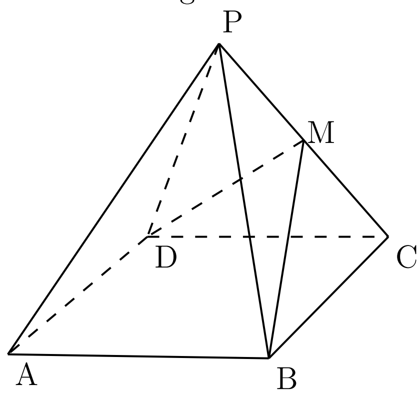


Without loss of generality, let the edge-length of $ABCD$ be 2. It follows that $CM = DM = \sqrt{3}$.
By the Law of Cosines,

$$\cos(\angle CMD) = \frac{CM^2 + DM^2 - CD^2}{2(CM)(DM)} = \boxed{\textbf{(B) } \frac{1}{3}}.$$

1.2 Example

In the diagram, $PABCD$ is a pyramid with square base $ABCD$ and with $PA = PB = PC = PD$. Suppose that M is the midpoint of PC and that $\angle BMD = 90^\circ$. Triangular-based pyramid $MBCD$ is removed by cutting along the triangle defined by the points M, B and D. The volume of the remaining solid $PABMD$ is 288. What is the length of AB?



Let the side length of the square base $ABCD$ be $2a$ and the height of the pyramid (that is, the distance of P above the base) be $2h$.
Let F be the point of intersection of the diagonals AC and BD of the base. By symmetry, P is directly above F; that is, PF is perpendicular to the plane of square $ABCD$.
Note that $AB = BC = CD = DA = 2a$ and $PF = 2h$.
We want to determine the value of $2a$.
Let G be the midpoint of FC.
Join P to F and M to G.
Consider $\triangle PCF$ and $\triangle MCG$.
Since M is the midpoint of PC, then $MC = \frac{1}{2}PC$.
Since G is the midpoint of FC, then $GC = \frac{1}{2}FC$.
Since $\triangle PCF$ and $\triangle MCG$ share an angle at C and the two pairs of corresponding sides adjacent to this angle are in the same ratio, then $\triangle PCF$ is similar to $\triangle MCG$.
Since PF is perpendicular to FC, then MG is perpendicular to GC.
Also, $MG = \frac{1}{2}PF = h$ since the side lengths of $\triangle MCG$ are half those of $\triangle PCF$.
The volume of the square-based pyramid $PABCD$ equals $\frac{1}{3}\left(AB^2\right)(PF) = \frac{1}{3}(2a)^2(2h) = \frac{8}{3}a^2h$.
Triangular-based pyramid $MBCD$ can be viewed as having right-angled $\triangle BCD$ as its base and MG as its height.
Thus, its volume equals $\frac{1}{3}\left(\frac{1}{2}\cdot BC\cdot CD\right)(MG) = \frac{1}{6}(2a)^2h = \frac{2}{3}a^2h$.
Therefore, the volume of solid $PABMD$, in terms of a and h, equals $\frac{8}{3}a^2h - \frac{2}{3}a^2h = 2a^2h$.
Since the volume of PABMD is 288 , then $2a^2h = 288$ or $a^2h = 144$.
We have not yet used the information that $\angle BMD = 90^\circ$.
Since $\angle BMD = 90^\circ$, then $\triangle BMD$ is right-angled at M and so $BD^2 = BM^2 + MD^2$.
By symmetry, $BM = MD$ and so $BD^2 = 2BM^2$.
Since $\triangle BCD$ is right-angled at C, then $BD^2 = BC^2 + CD^2 = 2(2a)^2 = 8a^2$.
Since $\triangle BGM$ is right-angled at G, then $BM^2 = BG^2 + MG^2 = BG^2 + h^2$.

Since $\triangle BFG$ is right-angled at F (the diagonals of square $ABCD$ are equal and perpendicular), then

$$\begin{aligned} BG^2 = BF^2 + FG^2 &= \left(\frac{1}{2}BD\right)^2 + \left(\frac{1}{4}AC\right)^2 = \frac{1}{4}BD^2 + \frac{1}{16}AC^2 \\ &= \frac{1}{4}BD^2 + \frac{1}{16}BD^2 = \frac{5}{16}BD^2 = \frac{5}{2}a^2 \end{aligned}$$

Since $2BM^2 = BD^2$, then $2\left(BG^2 + h^2\right) = 8a^2$ which gives $\frac{5}{2}a^2 + h^2 = 4a^2$ or $h^2 = \frac{3}{2}a^2$ or $a^2 = \frac{2}{3}h^2$.
Since $a^2h = 144$, then $\frac{2}{3}h^2 \cdot h = 144$ or $h^3 = 216$ which gives $h = 6$.
From $a^2h = 144$, we obtain $6a^2 = 144$ or $a^2 = 24$.
Since $a > 0$, then $a = 2\sqrt{6}$ and so $AB = 2a = 4\sqrt{6}$.
ANSWER: $\boxed{4\sqrt{6}}$

1.3 Example

Three spheres with radii 11, 13, and 19 are mutually externally tangent. A plane intersects the spheres in three congruent circles centered at A, B, and C, respectively, and the centers of the spheres all lie on the same side of this plane. Suppose that $AB^2 = 560$. Find AC^2.

Denote by r the radius of three congruent circles formed by the cutting plane. Denote by O_A, O_B, O_C the centers of three spheres that intersect the plane to get circles centered at A, B, C, respectively. Because three spheres are mutually tangent, $O_A O_B = 11 + 13 = 24$, $O_A O_C = 11 + 19 = 30$. We have $O_A A^2 = 11^2 - r^2$, $O_B B^2 = 13^2 - r^2$, $O_C C^2 = 19^2 - r^2$. Because $O_A A$ and $O_B B$ are perpendicular to the plane, $O_A A B O_B$ is a right trapezoid, with $\angle O_A A B = \angle O_B B A = 90^\circ$. Hence,

$$\begin{aligned} O_B B - O_A A &= \sqrt{O_A O_B^2 - AB^2} \\ &= 4. \qquad (1) \end{aligned}$$

Recall that

$$\begin{aligned} O_B B^2 - O_A A^2 &= \left(13^2 - r^2\right) - \left(11^2 - r^2\right) \\ &= 48. \qquad (2) \end{aligned}$$

Hence, taking $\frac{(2)}{(1)}$, we get

$$O_B B + O_A A = 12. \qquad (3)$$

Solving (1) and (3), we get $O_B B = 8$ and $O_A A = 4$. Thus, $r^2 = 11^2 - O_A A^2 = 105$. Thus, $O_C C = \sqrt{19^2 - r^2} = 16$. Because $O_A A$ and $O_C C$ are perpendicular to the plane, $O_A A C O_C$ is a right trapezoid, with $\angle O_A A C = \angle O_C C A = 90^\circ$. Therefore,

$$\begin{aligned} AC^2 &= O_A O_C^2 - \left(O_C C - O_A A\right)^2 \\ &= \boxed{756}. \end{aligned}$$