

# Geometry 2 - Circles

TSS Math Club

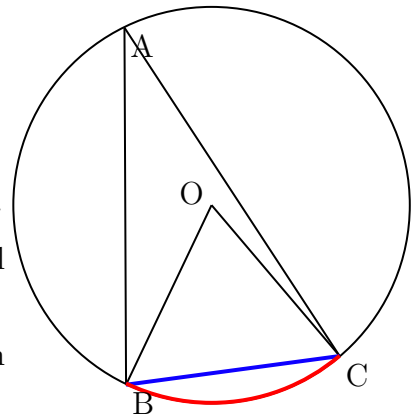
Oct 2022

## 1 Basic property of Circles

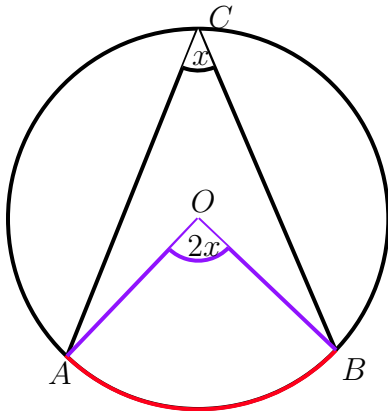
### 1.1 Definition of Circles

### 1.2 Terms to describe geometric object related to circles

- Center: Point O.
- Radius: Length from center to perimeter.
- Arc: A curved line on the circumference of a circle.
- Chord: A straight line between two points on a circle.
- Central angle:  $\angle BOC$  would be an example of a central angle.
- Inscribed angle:  $\angle BAC$  would be an example of an inscribed angle.



### 1.3 Central angle is twice any inscribed angle



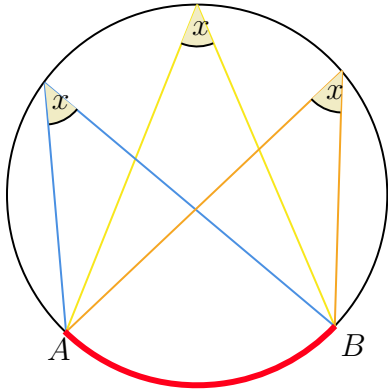
Create a line from point C that goes through point O and hits the circumference of the circle. Name this point P, and label  $\angle ACO$  as  $y$ .

Since triangle  $\triangle COA$  is an isosceles ( $CO = AO$ ),  $\angle ACO$  and  $\angle CAO$  are the same, and thus  $\angle AOP$  is equal to  $2y$ .

Label  $\angle BCO$  as  $z$ , and since  $\triangle COB$  is also an isosceles triangle,  $\angle BOC$  is equivalent to  $2z$ .

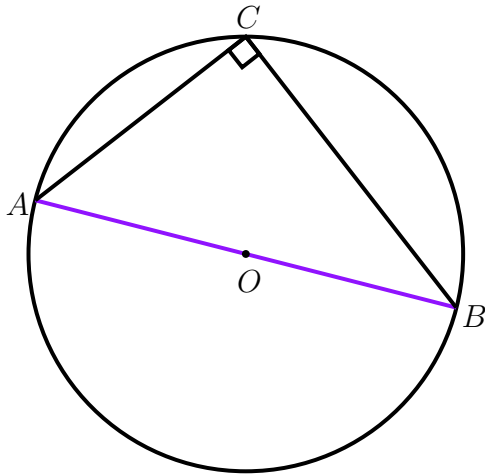
Therefore,  $\angle AOB = 2y + 2z = 2y + z = 2x$ .

#### 1.4 Inscribed angles subtended by the same arc are equal



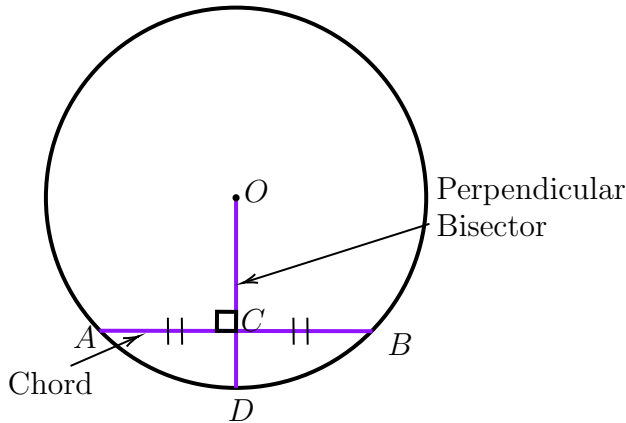
Since an inscribed angle is half of its center angle with the same arc (ref 1.3), and all three angles share the same arc, the inscribed angles are all equal.

#### 1.5 Angle subtended by a diameter is $90^\circ$



Both angles share the same arc, thus  $\angle ACB$  is half of  $\angle AOB$   $90^\circ$  is half of  $180^\circ$ .

## 1.6 Perpendicular chord theorem



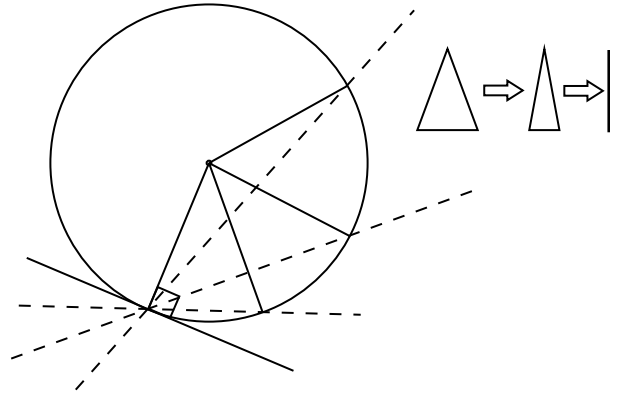
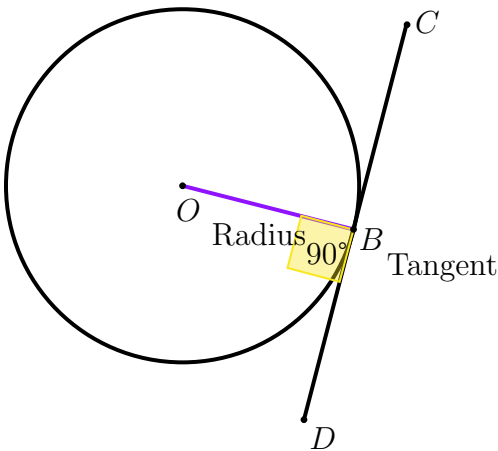
Connect point A to O, and point O to B. Note: AO and BO are both the radii of the circle. As a result,  $\triangle AOC$  and  $\triangle BOC$  are congruent triangles, with  $AC = BC$ . Therefore, the radius bisects AB.

## 1.7 Tangent to a circle

### 1.7.1 Definition

A line that intersects a circle at only one point.

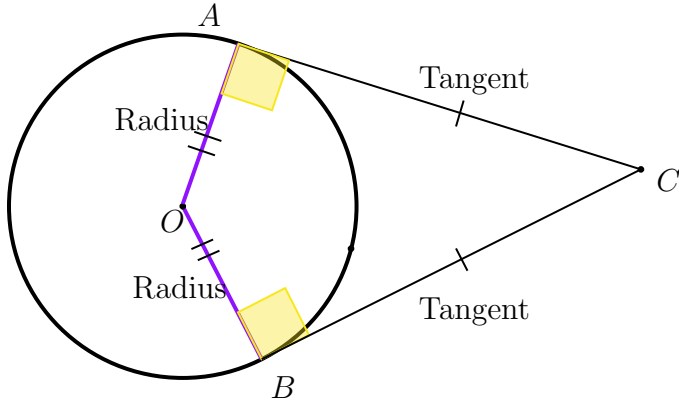
### 1.7.2 The radius from the center of the circle to the point of tangency is perpendicular to the tangent line



By having a line intersect a circle at random, a triangle is made when connecting the points of intersection to the center of the circle. The interior angle that is made can then be measured to be, presumably, less than  $90^\circ$  (when there are two points of intersection made).

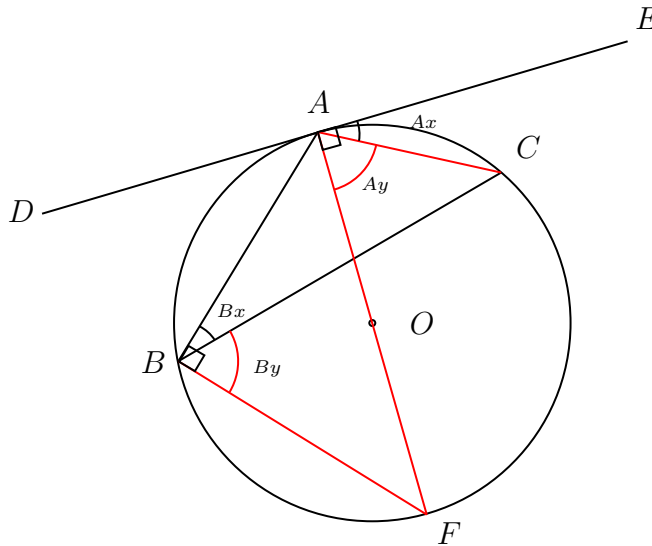
By bringing one of the points of intersection closer to the other, the triangle created becomes thinner and thinner, until it becomes a line that has an interior angle of  $90^\circ$ , and thus becomes tangent to the circle (intersects the circle at 1 point, and at a  $90^\circ$  angle).

**1.7.3 The length of tangents from a point to a circle are equal**



Since  $AO = BO$ ,  $OC$  is shared, and  $\angle OAC = \angle OBC$  (SSA),  $AC = BC$  (Hypotenuse Leg Theorem).

**1.7.4 Tangent-Chord Theorem: the angle formed between a chord and a tangent line to a circle is equal to the inscribed angle on the other side of the chord**



To prove that  $\angle EAC$  is equivalent to  $\angle ABC$ , draw a line from point A that passes through the center and intersects with the circumference of the circle. Since  $DE$  is tangent to the circle,  $\angle EAF$  is  $90^\circ$ .  $\angle CBF$  is also  $90^\circ$ , since  $\angle CBF$  shares the same arc as  $\angle EAF$  (ref. 1.4).

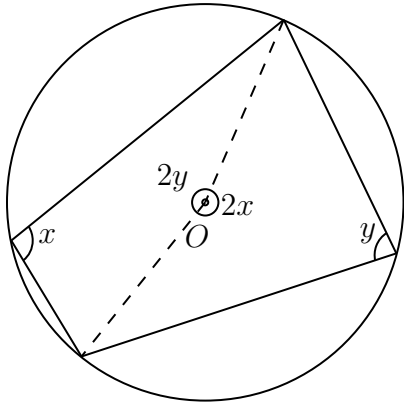
$\angle EAF$  can be considered as the sum of  $Ax$  and  $Ay$ . We know that  $By$  and  $Ay$  are the same angle because they share the same arc. Thus, we know  $Ax = Bx$ , since  $90^\circ - Ay = 90^\circ - By = Ax = Bx$ .

## 2 Cyclic Quadrilateral (Four points cyclic)

### 2.1 Definition

A quadrilateral which has all its four vertices lying on the perimeter of a circle.

### 2.2 Opposite angles are added up to $180^\circ$

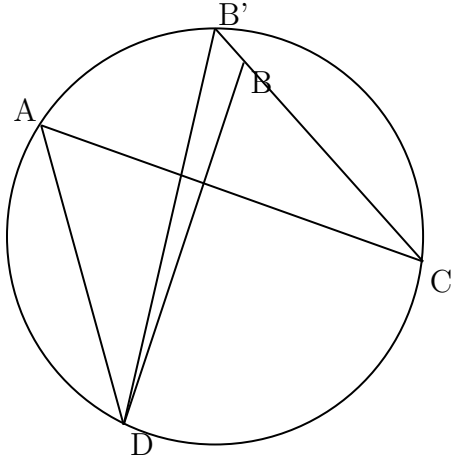


$$\begin{aligned} 2x + 2y &= 360^\circ \\ (x + y)/2 &= 360^\circ/2 \\ x + y &= 180^\circ \end{aligned}$$

### 2.3 How to prove four points cyclic

2.3.1 Prove these four points lie equally distant to another point — the center of the circle

2.3.2 Two equal angles subtend a segment (chord in the circle)



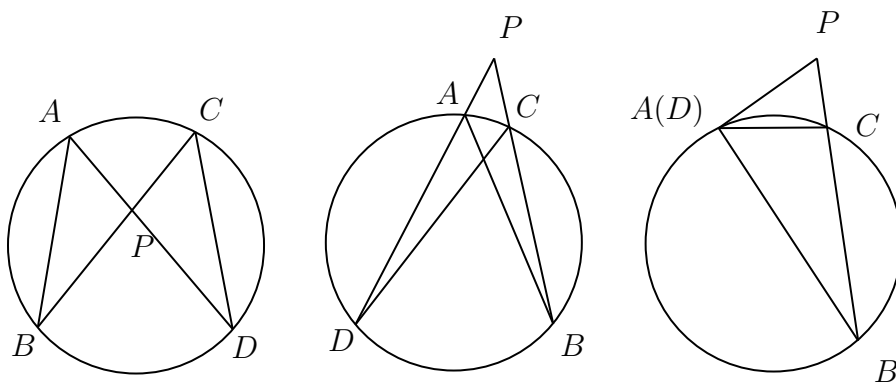
If not, then let BC intersect circle at B'.  
 By 1.4,  $\angle B' = \angle A$   
 By Hypothesis,  $\angle DBC = \angle A$   
 Then  $\angle BDB' = \angle B' - \angle DBC = 0$   
 Therefore, B and B' must be the same point.

2.3.3 The sum of the opposite angle is  $180^\circ$

Proof is similar to 2.3.2

### 3 Similar triangles involving a circle

#### 3.1 Identify as many similar triangles as possible

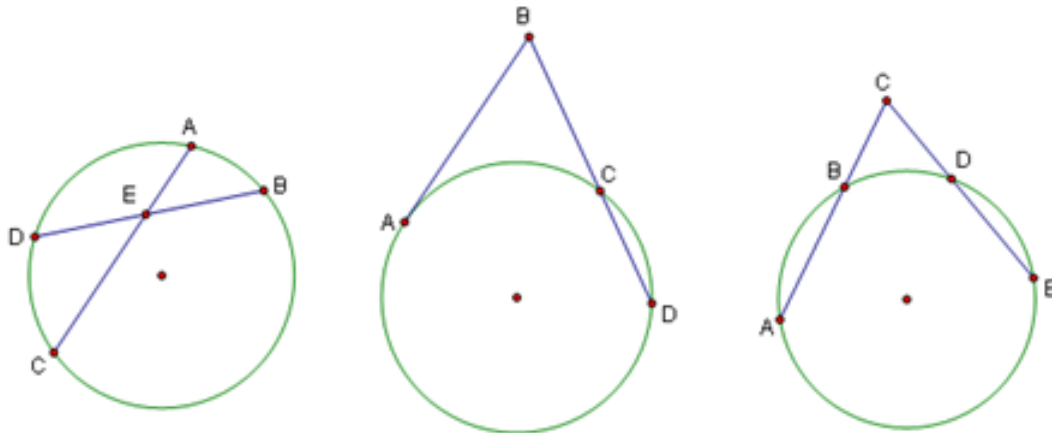


#### 3.2 Power of a point

##### 3.2.1 Definition:

A real number that demonstrates the distance a point from a circle.

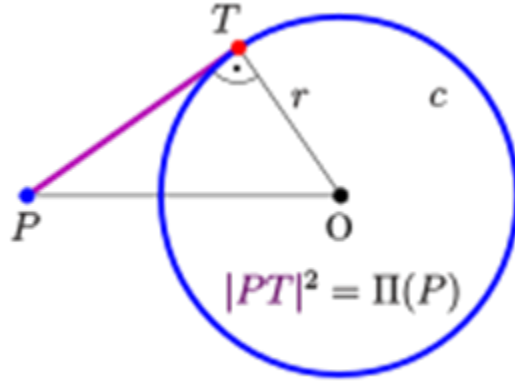
##### 3.2.2 Power of point is fixed regardless the choice of chord



A ratio between the side lengths of the two similar triangles DEC and AEB can be made by having a fraction with one line over it's "pair". Therefore, for the first figure,  $EA/DE = BE/CE \rightarrow EA \times EC = BE \times DE$ . As a result, the power of a point is the same for any two intersections made by a straight line from a single point. As a result, we can determine that  $BA \times BA = BC \times BD$ , and  $CB \times CA = CD \times CE$ , for figure 2 and 3 respectively.

### 3.2.3 Power of a point formula

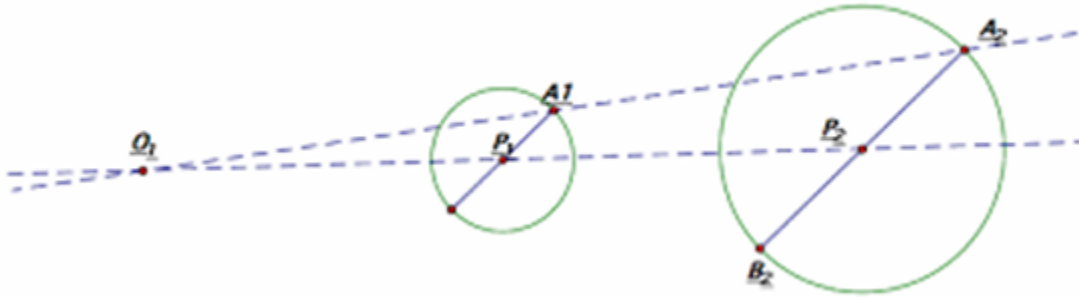
$$PoP = PO^2 - r^2$$



We know that  $PoP = PT^2$ , and that  $PT$  is tangent to the circle, therefore making  $\angle PTO = 90^\circ$ . We can then use the Pythagorean theorem, resulting in:  $PT^2 = PO^2 - r^2$ .

## 3.3 Homothety involving circles

### 3.3.1 Homothety of a circle is a circle



The ratio between the two circles can be displayed as a fraction:  $OP_1/OP_2$ . Therefore,  $P_1A_1 = R$ , and  $P_2A_2 = R \times r$ .

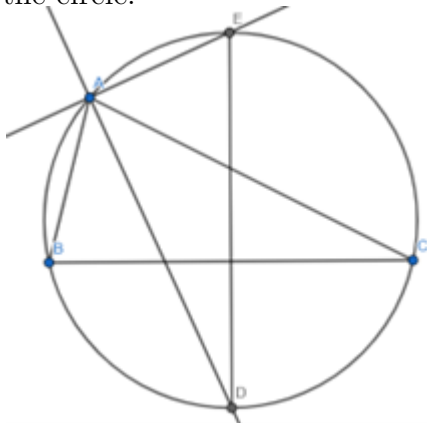
### 3.3.2 Ratios in the homothety

$R/r$  is the ratio of homothety (ref. 3.3.1).

## 4 Problems

### 4.1 Problem

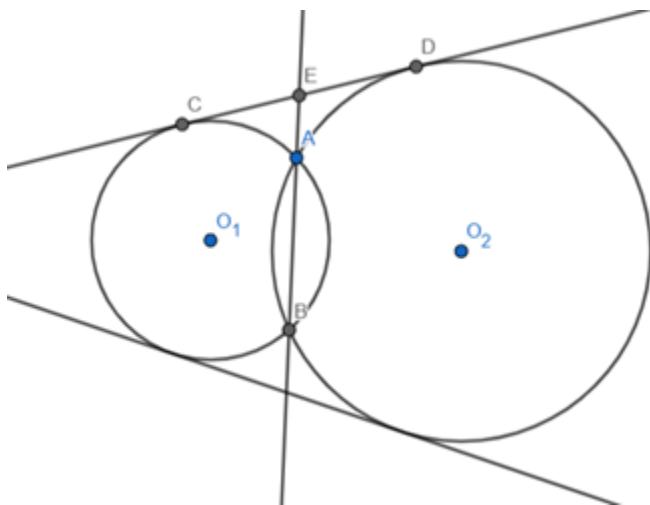
Given  $AD$   $AE$  are the internal, external angle bisector of angle  $A$ , such that  $D, E$  are the intersection of the angle bisectors with the circumcircle. Prove  $DE$  is a diameter of the circle.



Since  $EAD = 90^\circ$ ,  $DE$  is a diameter of the circumcircle. In addition, since  $AD$  is the internal angle bisector,  $D$  is the midpoint of arc  $AB$  as  $\angle BAD = \angle CAD$ . Similarly,  $E$  is the midpoint of arc  $BAC$ .

### 4.2 Problem

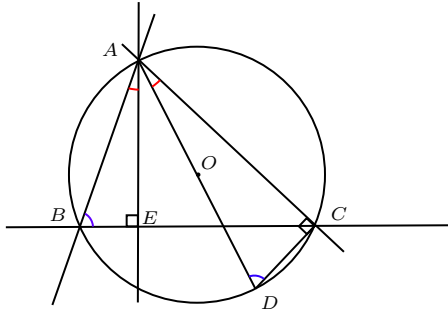
Given Circle  $C_1$ ,  $C_2$  intersect at  $A$ ,  $B$ ,  $CD$  is the common tangent to both circles,  $E$  is the intersection of  $AB$  and  $CD$ . Prove  $E$  is the midpoint of  $CD$ .



$$\begin{aligned} EC^2 &= EA \times EB \\ ED^2 &= EA \times EB \\ \text{Since, } EC^2 &= ED^2 \\ \text{Therefore, } EC &= ED \end{aligned}$$

### 4.3 Theorem

In a triangle  $abc=4RS$



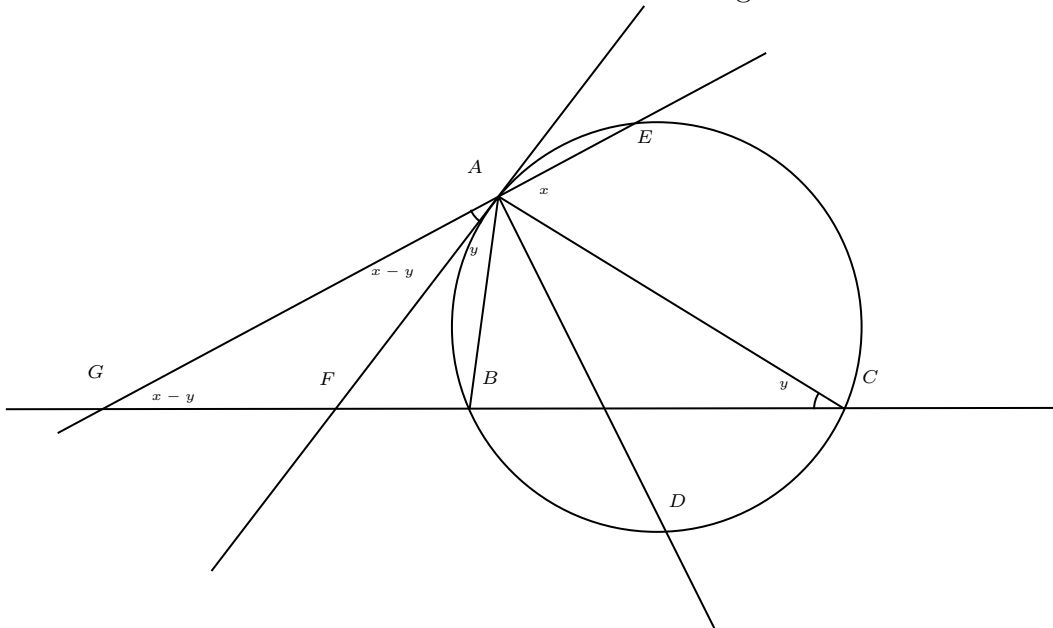
We know  $\triangle AEB$  is similar to  $\triangle ACD$ , since  $\angle BAE = \angle DAC$  ( $\angle ABC = \angle ADC$  since they share the same arc, and  $AE$  is perpendicular to  $EB$ , similar to how  $AC$  is perpendicular to  $DC$ ).

Therefore, we can state  $AE/AB = AC/AD \longrightarrow AE \times 2r = AC \times AB = bc$ .

Finally,  $abc = 2R \times AE \times a = 4R \times \left(\frac{AE \times BC}{2}\right) = 4RS$ , where S is the area of  $\triangle ABC$

## 4.4 Problem

Given AE is the external angle bisector of angle A, AE intersects BC at G, the tangent at A intersects BC at F. Prove AFG is an isosceles triangle.



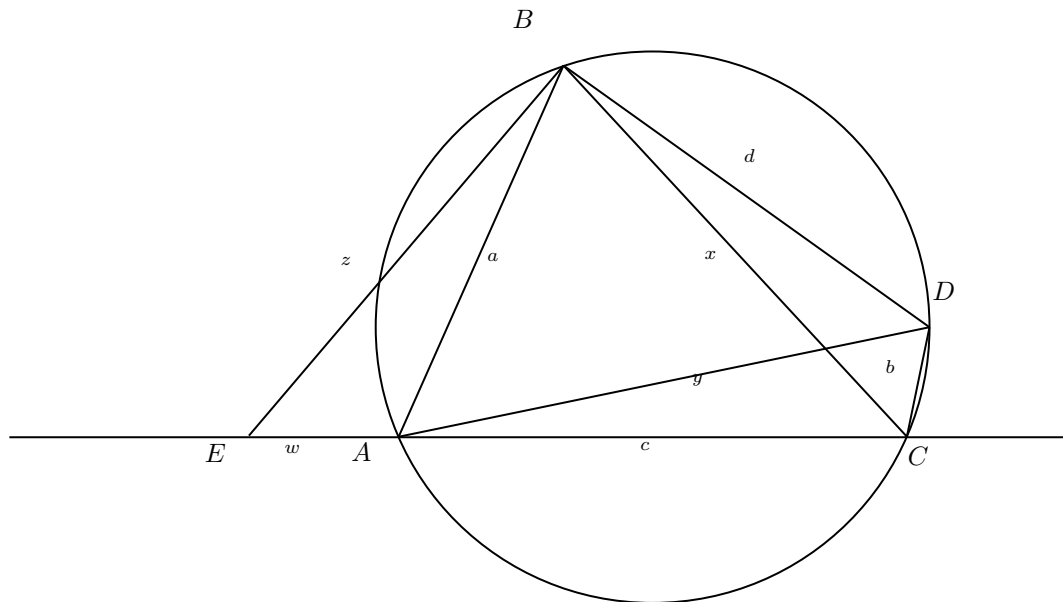
Since  $AE$  is the external angle bisector,  $\angle GAB = \angle EAC$ .

Also, by Tangent-Chord Theorem,  $\angle FAB = \angle ACB$ .

Therefore,  $\angle G = \angle EAC - \angle ACB = \angle GAB - \angle FAB = \angle GAF$  (External Angle Theorem), and thus, AFG is an isosceles triangle.

## 4.5 Ptolemy's theorem

If a quadrilateral is inscribed in a circle then the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides. Or  $ab+cd=xy$  where  $a,b,c,d$  are the sides of the quadrilateral and  $x,y$  are the diagonals.



First, draw a reference line between point B that intersects the extended line AC with the same angle as  $\angle DBC$  and label this point E. We know that  $\triangle BEA$  is similar to  $\triangle BCD$ , and since they both share  $\angle ABC$ ,  $\triangle EBC$  is similar to  $\triangle ABD$ . Thus, we can create equivalent ratios with the various side lengths available:

$$\frac{z}{x} = \frac{a}{d} = \frac{w}{b} \implies ab = wd$$

Since  $\triangle BEC$  is similar to  $\triangle BAD$ ,

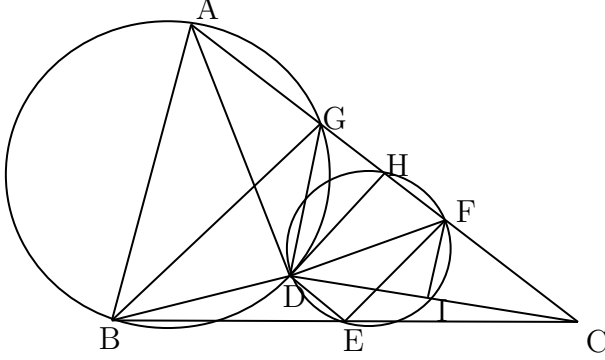
$$\frac{z}{a} = \frac{x}{d} = \frac{w+c}{y}$$

$$wd + cd = xy$$

$$ab + cd = xy$$

#### 4.6 Problem

In  $\triangle ABC$ , point D is inside of ABC such that  $\angle DAC = \angle DCA = 30^\circ$  and  $\angle DBA = 60^\circ$ . E is the midpoint on BC and F is a trisect point on AC such that  $CF = \frac{CA}{3}$ . Prove  $DE \perp EF$ .



Let G be the midpoint of AF, Draw DH perpendicular to AC, FI perpendicular to CD. Connect DG, DF, BG

Claim 1: DFG is an equilateral triangle and AGD and DFC are 30-120-30 triangles.

Proof of Claim 1:

Let  $DC = x$ .

Since  $\angle DAC = \angle DCA = 30^\circ$

$$DH = \frac{x}{2}, CH = \frac{\sqrt{3}x}{2}, HF = \frac{HC}{3}$$

$$\text{Therefore, } HF = \frac{\sqrt{3}x}{6} = \frac{DH}{\sqrt{3}}$$

This implies that DHF is a 30-60-90 triangle and DFC is a 30-120-30 triangle.

Similarly, AGD is a 30-120-30 triangle and therefore DFG is an equilateral triangle.

By Claim 1, AGDB concyclic since  $\angle ABD = \angle FGD = 60^\circ$

Also by Claim 1, I is the midpoint of CD.

Now, consider the homothety from C with ratio  $1/2$ .

It is clear that A maps to H, B maps to E, D maps to I, G maps to F. Therefore, Circle ABDG maps to the circle HEIF (Note we haven't proved that D also lies on the same circle).

Now, let's prove that D also lies on the same circle.

Since  $\angle FDI = 30^\circ$ , enough to show the arc  $IF = 60^\circ$  (arc is measured by the central angle).

However, by homothety, the measure of arc IF is the same as the measure of arc DG  $= 60^\circ$  as  $\angle DAG = 30^\circ$

Therefore, D lies in the circle HEIF or HDEIF concyclic.

Since HDEIF concyclic,  $\angle DEF = \angle DIF = 90^\circ$

Therefore,  $DE \perp EF$ .

Q.E.D.