Combinatorics

TSS Math Club

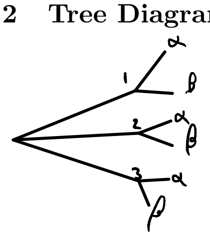
March 2023

1 Introduction to Counting

1.1 Rule of Product

1.1.1 Definition

1.1.2 Tree Diagram



1.1.3 Example 1

How many 4 digit numbers are there with no repeated digits?

$9 \times 9 \times 8 \times 7 =$

1.1.4 Example 2

How many 5 digit odd numbers are there with no repeated digits?

$8 \times 8 \times 7 \times 6 \quad \times 5 = 13440$

8 8 7 6 5

1 3 5 7 9

1.1.5 Example 3

How many ways are there for 5 people to stand in a row?

$5!$

1.2 Rule of Sum

1.2.1 Definition

1.2.2 Example 1

Calvin wants to go to Milwaukee. He can choose from 3 bus services or 3 train services to head from home to downtown Chicago. From there, he can choose from 2 bus services or 3 train services to head to Milwaukee. How many ways are there for Calvin to get to Milwaukee?

$6 \times 5 = 30$

1.3 Case Working

1.3.1 Definition

If you cannot solve a question in "one shot", you can consider all the cases and add the values.

1.3.2 Example 1

How many numbers less than 10,000 are there with no repeated digits?

$9\times9\times8\times7 + 9\times9\times8 + 9\times9 + 9$

1.3.3 Example 2

Consider the flag:

If Q1 and 3 are same

10	1
9	9

Q1 and 3 are not

10	8
9	8

If 10 colours are provided, how many ways are there to colour the flag so that no 2 adjacent regions have the same colour.

$10\times9\times9 + 10\times9\times8\times8 = 6570$

1.4 Indirect/Complementary Counting

1.4.1 Example 1

How many positive integers less than 100 are not a multiple of five?

$\underbrace{99}_{\text{\# possible}} - \underbrace{19}_{\text{there are multiple of 5}} = 80$

1.4.2 Example 2

How many four-digit positive integers have at least one digit that is a 2 or a 3?

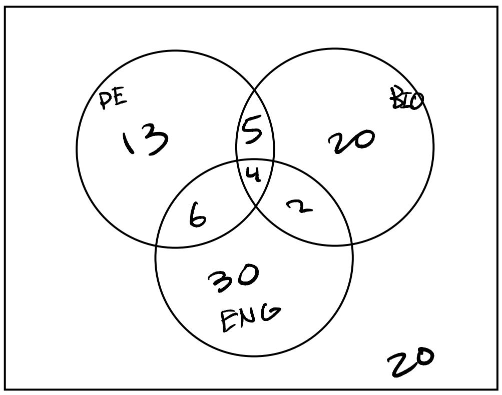
$\underbrace{9\times10\times10\times10}_{\text{\# of possible values}} - \underbrace{7\times8\times8\times8}_{\text{\# of values without a 2 or a 3.}} = 5416$

1.5 Venn Diagram and Inclusion-Exclusion Principle

1.5.1 Venn Diagram

Example: 100 students were interviewed. 28 took PE, 31 took BIO, 42 took ENG, 9 took PE and BIO, 10 took PE and ENG, 6 took BIO and ENG, 4 took all three subjects.

- How many students took none of the three subjects? $=20$
- How many students took PE but not BIO or ENG? $=13$
- How many students took BIO and PE but not ENG? $=5$



1.5.2 Inclusion-Exclusion Principle

Two sets: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.
Three sets: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$.

Example:
There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

$\because \quad |A_1| = 10$

$|A_2| = 13$

$|A_3| = 9$

$|A_1 \cup A_2 \cup A_3| = 20$

$|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1| - 2|A_1 \cap A_2 \cap A_3| = 9$

$\therefore \quad 20 = 10 + 13 + 9 - 9 - |A_1 \cap A_2 \cap A_3|$

$20 = 23 - |A_1 \cap A_2 \cap A_3|$

$3 = |A_1 \cap A_2 \cap A_3|$

2 Permutation and Combination

2.1 Factorial

2.1.1 Notation/Definition

$n! = n \times (n-1) \times (n-2) \times \dots \times 2 \times 1$

2.1.2 Examples

- $3! = 3 \times 2 \times 1 = 6$
- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
- $(n^2 + 5n + 6)(n+1)! = (n+3)!$

2.2 Permutation

2.2.1 Definition

m objects, put into n slots.

m

$\_ \ \_ \ \_ \ \_ \quad n$

2.2.2 Notation

$$mPn = P(m, n) = \frac{m!}{(m-n)!}$$

2.2.3 Examples

- $5P3 = \frac{5!}{(5-3)!} = \frac{5!}{2!}$
- $P(7, 2) = \frac{7!}{(7-2)!} = \frac{7!}{5!}$
- $P(n, r)P(n-r, m) = \frac{n!}{(n-r)!} \times \frac{(n-r)!}{(n-r-m)!} = \frac{n!}{(n-r-m)!}$

2.3 With Repetition

2.3.1 No repetiotion

How many arrangements are there for the word MATHS?

$5!$

2.3.2 With repetiotion

How many arrangements are there for the word TORONTO?

$$\frac{7!}{3!2!}$$

2.3.3 Problem

How many ways can 12 basketball players be assigned to four triple rooms?

$$\frac{12!}{(3!)^4}$$

2.4 Grouping

2.4.1 Example

Math Club is taking yearbook photo one day with 20 members and Mr.Fraschetti and Mr.Gatti lining in a row. If Mr.Fraschetti and Mr.Gatti insist to stand together, how many arrangements are there?

$21! \times 2$

2.4.2 Problem

At a recent conference of the 11 premiers (including the Prime Minister), find the number of different group photos possible if Lucien Bouchard and Jean Chretien refused to stand next to each other and if the premiers arranged themselves in a line.

$11! - 10! \times 2$

2.5 Circular

2.5.1 Example

How many arrangements are there if 8 people sitting around a circular table?

$\frac{8!}{8} = 7!$

2.5.2 Problem

Grace is making a bracelet with 8 beads that are all different colours. How many bracelets can she make if the bracelet has no visible clasp?

$\frac{8!}{8 \times 2} = \frac{7!}{2}$

2.6 Combination

2.6.1 Definition

you have m objects, and you want to choose n of them

2.6.2 Notation

$mCn = \binom{m}{n} =$ m choose n : m # of units, n units to choose. $\binom{m}{n} = \frac{m!}{(n!)(m-n)!}$

2.6.3 Examples

- $5C3 = \frac{5!}{3!\,2!}$
- $\binom{7}{2} = \frac{7!}{2!\,5!}$
- Prove $\binom{n}{r} = \binom{n}{n-r}$

$LS = \frac{n!}{r!(n-r)!}$

$RS = \frac{n!}{(n-r)!(n-\cancel{n}+r)!}$

$= \frac{n!}{(n-r)!\,r!}$

$\therefore LS = RS$

2.7 Combination Problems

2.7.1 Problem

Find the number of different five-card hands that could be dealt from a deck of 52 cards. $\binom{52}{5} = \frac{52!}{5!\,47!}$

2.7.2 Problem

From a group of 14 Conservatives, 12 Litkrals, eight NDP, and two Independent Members of Parliament, how many different committees can be formed consisting of three Conservatives, three Liberals, two NDP, and one Independent member?

$= \binom{14}{3}\binom{12}{3}\binom{8}{2}\binom{2}{1}$

2.7.3 Problem

How many divisors of 4200 are there? add 1 to the power to add the option of 0 being selected.

$4200 = 2^3 \times 3 \times 5^2 \times 7 \longrightarrow 4 \times 2 \times 3 \times 2 = 48$ options

2.8 Stars and Bars

2.8.1 Example

Find positive integers $\underbrace{m, n, k}_{3-1=2}$ such that $m + n + k = \underbrace{20}_{}$. , # of bars # of stars

$\overbrace{\star|\star|\star\,\star \ldots \star}^{20}$

$\therefore \binom{19}{2} = \frac{19!}{2!\,17!}$

2.8.2 Example

Find non-negative integers m, n, k such that $m + n + k = 20$.

Method 1

$(m+1)+(n+1)+(k+1) = 23$

$m' + n' + k' = 23$

$\therefore \binom{22}{2}$

Method 2

add 3 stars in beginning, remove 3 stars at the end (1 from each "basket")

$\therefore \binom{23}{2}$

2.8.3 Problem

If one wishes to count the number of ways to distribute seven indistinguishable one dollar coins among Amber, Ben, and Curtis so that each of them receives at least one dollar.

$\binom{6}{2}$

3 Pascal Triangle

3.1 Pascal Triangle

(Hockey Stick Theorem)

$n=0$							1						
$n=1$						1		1					
$n=2$					1		2		1				
$n=3$				1		3		3		1			
$n=4$			1		4		6		4		1		
$n=5$		1		5		10		10		5		1	
$n=6$	1		6		15		20		15		6		1

3.1.1 Pascal's Triangle Construction

Above 2 values produce next value (sum).

3.1.2 The Fundamental Relationship and Combination

$$\binom{n}{r}+\binom{n}{r+1} = \frac{n!}{r!(n-r)!}+\frac{n!}{(r+1)!(n-r-1)!}$$
$$= \binom{n+1}{r+1}$$

3.1.3 Addition of the Rows

2^n

3.1.4 Hockey Stick Theorem

Proof: move a "1" down a row, add adjacent values.

3.2 Binomial Theorem

3.2.1 Observation

Expand/FOIL $(a+b)^1$, $(a+b)^2$, $(a+b)^3$.

$(a+b)^2 = a^2+2ab+b^2$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

1, 3, 3, 1

Ref. Pascal's Triangle: $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

1, 4, 6, 4, 1

3.2.2 Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof:

Consider coeffients of each term in $(a+b)^n$ separately

for $\underbrace{a^{n-k} b^k}_{\text{the sum must be } n}$ $\qquad \underbrace{(a+b)(a+b) \cdots (a+b)}$

we need k b's from n brackets $\Rightarrow \binom{n}{k}$

3.2.3 Proof of Addition of the Rows

$(1+1)^n$

3.3 Generating Function

3.3.1 Example

Prove:

$$\sum_{i+j=n} \binom{m}{i} \binom{k}{j} = \binom{m+k}{n}$$

3.3.1. $\sum_{i+j=n} \binom{m}{i}\binom{k}{j} = \binom{m+k}{n}$

Consider $(1+x)^m (1+x)^k = (1+x)^{m+k}$

use binomal thm

$\Rightarrow \left(\binom{m}{0} + \binom{m}{1}x + \cdots \binom{m}{m}x^m\right)\left(\binom{k}{0} + \binom{k}{1}x + \cdots \binom{k}{k}x^k\right) = \left(\binom{m+k}{0} + \binom{m+k}{1}x + \cdots + \binom{m+k}{m+k}x^{m+k}\right)$

now focus on x^n

$\Rightarrow$ coefficient of x^n on RHS is $\binom{m+k}{n}$

what about LHS ?

$\left(\binom{m}{0} + \binom{m}{1}x + \cdots \binom{m}{m}x^m\right)\left(\binom{k}{0} + \binom{k}{1}x + \cdots + \binom{k}{n-1}x^{n-1} + \binom{k}{n}x^n + \cdots + \binom{k}{k}x^k\right)$

added together

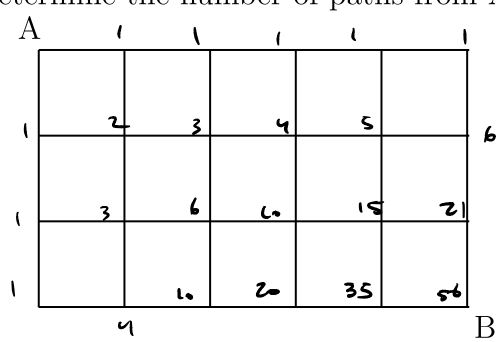
$\Rightarrow \binom{m}{0}\binom{k}{n} + \binom{m}{1}\binom{k}{n-1} + \cdots \binom{m}{n}\binom{k}{0}$

As LHS = RHS , $\sum_{i+j=n} \binom{m}{i}\binom{k}{j} = \binom{m+k}{n}$

3.4 Pascal Routes

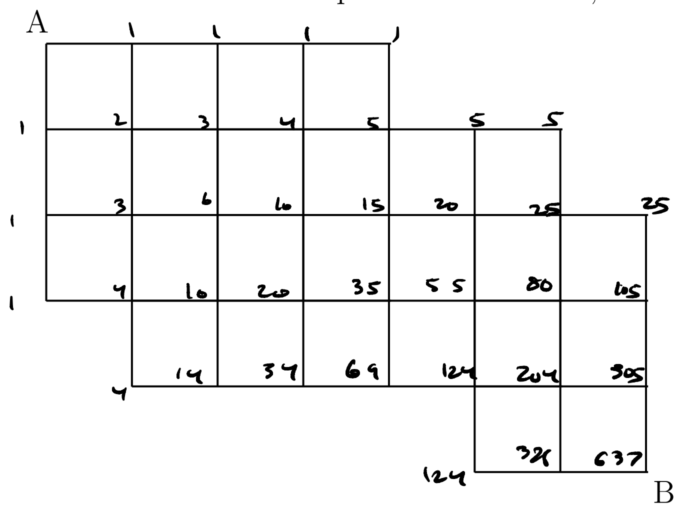
3.4.1 Example

Determine the number of paths from A to B, traveling downward and to the right.



3.4.2 Problem

Determine the number of paths from A to B, traveling downward and to the right.



4 Probability

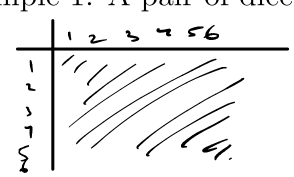
4.1 Definition

The probability of an event is a number that indicates how likely the event is to occur.
Mathematical Definition:

$$P(A) = \frac{\text{Number of times A happens}}{\text{Sample Space}}$$

4.1.1 Discrete

- Example 1: A pair of dice are rolled, what is the probability of getting a sum of 7?



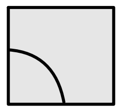
$\frac{6}{36} = 1/6$

- Example 2: 6 coins are tossed, what is the probability of getting a 2 head?

$= \frac{\binom{6}{2}}{2^6}$

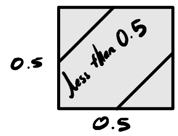
4.1.2 Continuous

- Example 1: Point A is chosen randomly in $[0, 2] \times [0, 2]$, what's the probability that OA is less than 1.?



$$\frac{\frac{\pi}{4}}{2\times 2} = \frac{\pi}{16}$$

- Example 2: 2 points are chosen randomly in [0,1], what's the probability that their distance is less than 0.5.



$\longrightarrow 3/4$

- Example 3: Point A is chosen randomly in $[0, 1]$, what's the probability that A is 0.25.?

$\because$ Point = length of 0

$\therefore$ $P = \frac{0}{1} = 0$

4.2 Some Formulae about Probability

- $P(E \cap F) = P(E)P(F)$ if E and F are independent events.
- $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
- $P(E|F) = \frac{P(E \cap F)}{P(F)}$.

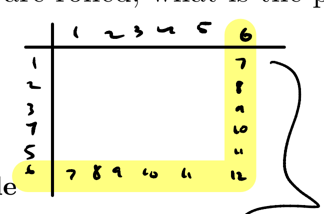
4.2.1 Example

What's the probability of tossing 3 heads in a row?

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

4.2.2 Example

A pair of dice are rolled, what is the probability of getting a sum of 7 given that one of them is 6?



11 options

$\therefore \frac{2}{11}$

4.2.3 Example

An integer is randomly chosen in [1,100], what is the probability that it is a multiple of 2 or 5?

mtp. 2 = 1/2 = 50
mtp. 5 = 1/5 = 20 } 70

$\underbrace{70 - 10}_{}$ = 60

both mtp. 5 and 2 or double counted values

$\therefore$ 60%

4.3 Expected Value

$$E = \sum \text{outcome} \times P(\text{outcome})$$

4.3.1 Example

A local club plans to invest \$10000 to host a baseball game. They expect to sell tickets worth \$15000 . But if it rains on the day of game, they won't sell any tickets and the club will lose all the money invested. If the weather forecast for the day of game is 20% possibility of rain, is this a good investment?

$EV = 15000 \times 80\% + 0 \times 20\%$
$= 12000$

$\because 12000 > 10000$, $\therefore$ good investment

4.3.2 Example

If Adam rolls a die repeatedly until he obtains a 6, what is the anticipated number of times he will need to roll the die?

$E = 1/6 + 2 \times \frac{5}{6} \times \frac{1}{6} + 3 \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \ldots$

$E = \frac{q}{p} + 1 = \frac{5/6}{1/6} + 1 = 6$

$\therefore$ He will need to roll the die 6 times to obtain a 6 on average.