

Combinatorics

TSS Math Club

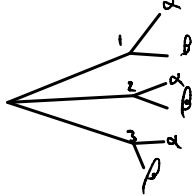
March 2023

1 Introduction to Counting

1.1 Rule of Product

1.1.1 Definition

1.1.2 Tree Diagram



1.1.3 Example 1

How many 4 digit numbers are there with no repeated digits?

$$9 \times 9 \times 8 \times 7 =$$

1.1.4 Example 2

How many 5 digit odd numbers are there with no repeated digits?

$$8 \times 8 \times 7 \times 6 \times 5 = 13440$$

$$\begin{array}{r} 8 \\ \hline 8 \\ \hline 7 \\ \hline 6 \\ \hline 5 \\ \hline 1 \\ \hline 1 \\ \hline 5 \\ \hline 2 \\ \hline 9 \end{array}$$

1.1.5 Example 3

How many ways are there for 5 people to stand in a row?

$$5!$$

1.2 Rule of Sum

1.2.1 Definition

1.2.2 Example 1

Calvin wants to go to Milwaukee. He can choose from 3 bus services or 3 train services to head from home to downtown Chicago. From there, he can choose from 2 bus services or 3 train services to head to Milwaukee. How many ways are there for Calvin to get to Milwaukee?

$$6 \times 5 = 30$$

1.3 Case Working

1.3.1 Definition

If you cannot solve a question in "one shot", you can consider all the cases and add the values.

1.3.2 Example 1

How many numbers less than 10,000 are there with no repeated digits?

$$9 \times 9 \times 8 \times 7 + 9 \times 9 \times 8 + 9 \times 9 + 9$$

1.3.3 Example 2

Consider the flag:

If Q1 and 3 are same

Q1 and 3 are not

10	1
9	9

10	8
9	8

If 10 colours are provided, how many ways are there to colour the flag so that no 2 adjacent regions have the same colour.

$$10 \times 9 \times 9 + 10 \times 9 \times 8 \times 8 = 6570$$

1.4 Indirect/Complementary Counting

1.4.1 Example 1

How many positive integers less than 100 are not a multiple of five?

$$99 - 19 = 80$$

possible \uparrow \uparrow there are

1.4.2 Example 2 multiple of 5

How many four-digit positive integers have at least one digit that is a 2 or a 3?

$$9 \times 10 \times 10 \times 10 - 7 \times 8 \times 8 \times 8 = 5416$$

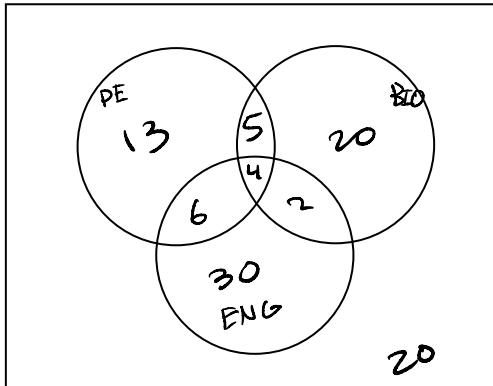
of possible values \uparrow # of values without a 2 or a 3

1.5 Venn Diagram and Inclusion-Exclusion Principle

1.5.1 Venn Diagram

Example: 100 students were interviewed. 28 took PE, 31 took BIO, 42 took ENG, 9 took PE and BIO, 10 took PE and ENG, 6 took BIO and ENG, 4 took all three subjects.

- How many students took none of the three subjects? ≈ 20
- How many students took PE but not BIO or ENG? ≈ 13
- How many students took BIO and PE but not ENG? ≈ 5



1.5.2 Inclusion-Exclusion Principle

Two sets: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

Three sets: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$.

Example:

There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

$$\therefore |A_1| = 10$$

$$|A_2| = 13$$

$$|A_3| = 9$$

$$|A_1 \cup A_2 \cup A_3| = 20$$

$$|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1| - 2|A_1 \cap A_2 \cap A_3| = 9$$

$$\therefore 20 = 10 + 13 + 9 - 9 - |A_1 \cap A_2 \cap A_3|$$

$$20 = 23 - |A_1 \cap A_2 \cap A_3|$$

$$3 = |A_1 \cap A_2 \cap A_3|$$

2 Permutation and Combination

2.1 Factorial

2.1.1 Notation/Definition

$$n! = n \times (n-1) \times (n-2) \times \dots \times 2 \times 1$$

2.1.2 Examples

- $3! = 3 \times 2 \times 1 = 6$
- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
- $(n^2 + 5n + 6)(n + 1)! = (n+3)!$

2.2 Permutation

2.2.1 Definition

n objects, put into n slots.



2.2.2 Notation

$${}_m P_n = P(m, n) = \frac{n!}{(n-m)!}$$

2.2.3 Examples

- ${}_5 P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!}$
- $P(7, 2) = \frac{7!}{(7-2)!} = \frac{7!}{5!}$
- $P(n, r)P(n-r, m) = \frac{n!}{(n-r)!} \times \frac{(n-r)!}{(n-r-m)!} = \frac{n!}{(n-r-m)!}$

2.3 With Repetition

2.3.1 No repetition

How many arrangements are there for the word MATHS?

$$5!$$

2.3.2 With repetition

How many arrangements are there for the word TORONTO?

$$\frac{7!}{3!2!}$$

2.3.3 Problem

How many ways can 12 basketball players be assigned to four triple rooms?

$$\frac{12!}{(3!)^4}$$

2.4 Grouping

2.4.1 Example

Math Club is taking yearbook photo one day with 20 members and Mr. Frascchetti and Mr. Gatti lining in a row. If Mr. Frascchetti and Mr. Gatti insist to stand together, how many arrangements are there?

$$21! \times 2$$

2.4.2 Problem

At a recent conference of the 11 premiers (including the Prime Minister), find the number of different group photos possible if Lucien Bouchard and Jean Chretien refused to stand next to each other and if the premiers arranged themselves in a line.

$$11! - 10! \times 2$$

2.5 Circular

2.5.1 Example

How many arrangements are there if 8 people sitting around a circular table?

$$\frac{8!}{8} = 7!$$

2.5.2 Problem

Grace is making a bracelet with 8 beads that are all different colours. How many bracelets can she make if the bracelet has no visible clasp?

$$\frac{8!}{8 \times 2} = \frac{7!}{2}$$

2.6 Combination

2.6.1 Definition

you have m objects, and you want to choose n of them

2.6.2 Notation

$mCn = \binom{m}{n} = m \text{ choose } n : m \text{ \# of units, } n \text{ units to choose.}$ $\binom{m}{n} = \frac{m!}{(n!)(m-n)!}$

2.6.3 Examples

- $5C3 = \frac{5!}{3!2!}$
- $\binom{7}{2} = \frac{7!}{2!5!}$
- Prove $\binom{n}{r} = \binom{n}{n-r}$

$$Ls = \frac{n!}{r!(n-r)!}$$

$$\begin{aligned} Rs &= \frac{n!}{(n-r)!(n-r+r)!} \\ &= \frac{n!}{(n-r)!r!} \end{aligned}$$

$$\therefore Ls = Rs$$

2.7 Combination Problems

2.7.1 Problem

Find the number of different five-card hands that could be dealt from a deck of 52 cards.

$$\binom{52}{5} = \frac{52!}{5! 47!}$$

2.7.2 Problem

From a group of 14 Conservatives, 12 Liberals, eight NDP, and two Independent Members of Parliament, how many different committees can be formed consisting of three Conservatives, three Liberals, two NDP, and one Independent member?

$$= \binom{14}{3} \binom{12}{3} \binom{8}{2} \binom{2}{1}$$

2.7.3 Problem

How many divisors of 4200 are there? *add 1 to the power to add the option of 0 being selected.*

$$4200 = 2^3 \times 3 \times 5^2 \times 7 \longrightarrow 4 \times 2 \times 3 \times 2 = 48 \text{ options}$$

2.8 Stars and Bars

2.8.1 Example

Find positive integers m, n, k such that $m + n + k = 20$.

$$\begin{array}{c} \overbrace{\star | \star | \star \dots \star}^{20} \\ \underbrace{\quad \quad \quad}_{3-1=2, \# \text{ of bars}} \quad \quad \quad \underbrace{\quad \quad \quad}_{\# \text{ of stars}} \end{array}$$

$$\therefore \binom{19}{2} = \frac{19!}{2! 17!}$$

2.8.2 Example

Find non-negative integers m, n, k such that $m + n + k = 20$.

Method 1

$$(m+1) + (n+1) + (k+1) = 23$$

$$m' + n' + k' = 23$$

$$\therefore \binom{22}{2}$$

Method 2

add 3 stars in beginning, remove 3 stars at the end (1 from each "basket")

$$\therefore \binom{23}{2}$$

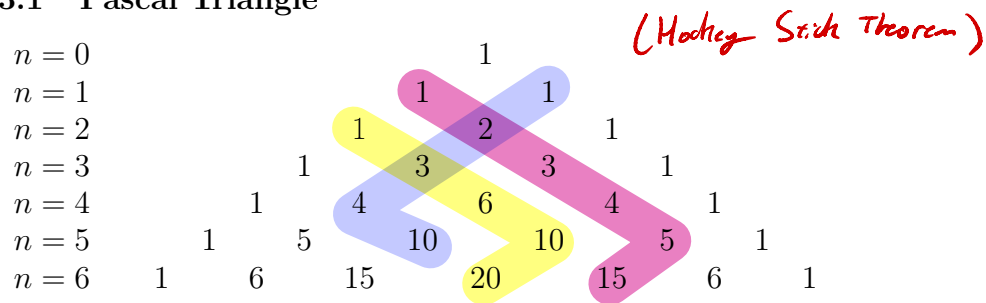
2.8.3 Problem

If one wishes to count the number of ways to distribute seven indistinguishable one dollar coins among Amber, Ben, and Curtis so that each of them receives at least one dollar.

$$\binom{6}{2}$$

3 Pascal Triangle

3.1 Pascal Triangle



3.1.1 Pascal's Triangle Construction

Above 2 values produce new value (sum).

3.1.2 The Fundamental Relationship and Combination

$$\binom{n}{r} + \binom{n}{r+1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!}$$

$$= \binom{n+1}{r+1}$$

3.1.3 Addition of the Rows

$$2^n$$

3.1.4 Hockey Stick Theorem

Proof: move a "1" down a row, add adjacent values.

3.2 Binomial Theorem

3.2.1 Observation

Expand/FOIL $(a+b)^1$, $(a+b)^2$, $(a+b)^3$.

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$1, 3, 3, 1$$

Ret. Pascal's Triangle: $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$$1, 4, 6, 4, 1$$

3.2.2 Binomial Theorem

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Proof:

Consider coefficients of each term in $(a+b)^n$ separately

for $a^{n-k} b^k$
the sum must be n

$(a+b)(a+b) \dots (a+b)$
we need k b's from n brackets $\Rightarrow \binom{n}{k}$

3.2.3 Proof of Addition of the Rows

$$(1+1)^n$$

3.3 Generating Function

3.3.1 Example

Prove:

$$\sum_{i+j=n} \binom{m}{i} \binom{k}{j} = \binom{m+k}{n}$$

$$\text{3.3.1. } \sum_{i+j=n} \binom{m}{i} \binom{k}{j} = \binom{m+k}{n}$$

$$\text{Consider } (1+x)^m (1+x)^k = (1+x)^{m+k}$$

use binomial theorem

$$\Rightarrow \left(\binom{m}{0} + \binom{m}{1}x + \dots + \binom{m}{n}x^n \right) \left(\binom{k}{0} + \binom{k}{1}x + \dots + \binom{k}{n}x^n \right) = \left(\binom{m+k}{0} + \binom{m+k}{1}x + \dots + \binom{m+k}{n+k}x^{n+k} \right)$$

now focus on x^n

\Rightarrow coefficient of x^n on RHS is $\binom{m+k}{n}$

what about LHS?

$$\left(\binom{m}{0} + \binom{m}{1}x + \dots + \binom{m}{n}x^n \right) \left(\binom{k}{0} + \binom{k}{1}x + \dots + \binom{k}{n-1}x^{n-1} + \binom{k}{n}x^n + \dots + \binom{k}{k}x^k \right)$$

added together

$$\Rightarrow \binom{m}{0}\binom{k}{n} + \binom{m}{1}\binom{k}{n-1} + \dots + \binom{m}{n}\binom{k}{0}$$

$$\text{As LHS = RHS, } \sum_{i+j=n} \binom{m}{i} \binom{k}{j} = \binom{m+k}{n}$$

3.4 Pascal Routes

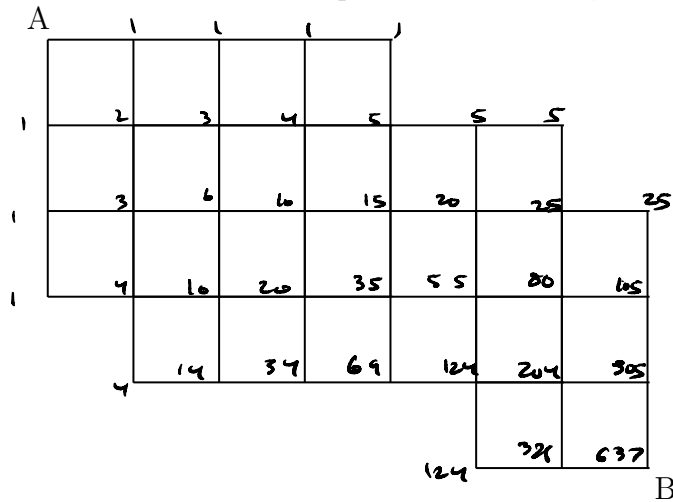
3.4.1 Example

Determine the number of paths from A to B, traveling downward and to the right.

A	1	1	1	1	1
1	2	3	4	5	6
1	3	6	10	15	21
1	4	10	20	35	56
	5	15	35	70	126
	6	21	56	126	252
	7	28	84	210	462
	8	36	126	350	840
	9	45	168	504	1360
	10	55	220	700	2100
	11	66	286	924	2860
	12	78	364	1176	3770
	13	91	455	1512	4860
	14	105	560	1944	6160
	15	120	680	2480	7740
	16	136	816	3136	9600
	17	153	969	3920	11760
	18	171	1140	4840	14200
	19	190	1330	5900	17000
	20	210	1540	7140	20160
	21	231	1771	8580	23760
	22	253	2024	10230	28000
	23	276	2300	12100	33000
	24	300	2600	14200	38800
	25	325	2925	16540	45400
	26	351	3276	19120	52800
	27	378	3654	21940	61000
	28	406	4060	25000	70000
	29	435	4495	28300	79800
	30	465	4960	31840	90400
	31	496	5456	35620	101800
	32	528	5984	39640	114000
	33	561	6546	43900	127000
	34	595	7144	48400	140800
	35	630	7779	53140	155400
	36	666	8452	58120	170800
	37	703	9164	63340	187000
	38	741	9916	68800	204000
	39	780	10710	74500	221800
	40	820	11548	80440	240400
	41	861	12432	86620	259800
	42	903	13364	93040	280000
	43	946	14346	99700	301000
	44	990	15380	106600	322800
	45	1035	16468	113740	345400
	46	1081	17612	121120	368800
	47	1128	18814	128740	393000
	48	1176	20076	136600	418000
	49	1225	21400	144700	443800
	50	1275	22788	153040	470400
	51	1326	24242	161620	497800
	52	1378	25764	170440	526000
	53	1431	27356	179500	555000
	54	1485	29020	188800	584800
	55	1540	30758	198340	615400
	56	1596	32572	208120	646800
	57	1653	34464	218140	679000
	58	1711	36436	228400	712000
	59	1770	38490	238900	745800
	60	1830	40628	249640	780400
	61	1891	42852	260620	815800
	62	1953	45164	271840	852000
	63	2016	47566	283300	889000
	64	2080	50060	295000	926800
	65	2145	52648	306940	965400
	66	2211	55332	319120	1004800
	67	2278	58114	331540	1045000
	68	2346	61006	344200	1086000
	69	2415	64010	357100	1127800
	70	2485	67128	370240	1170400
	71	2556	70362	383620	1213800
	72	2628	73714	397240	1258000
	73	2701	77186	411100	1303000
	74	2775	80780	425200	1348800
	75	2850	84498	439540	1395400
	76	2926	88342	454120	1442800
	77	3003	92314	468940	1491000
	78	3081	96416	484000	1539000
	79	3160	100650	499300	1587800
	80	3240	105018	514840	1637400
	81	3321	109522	530620	1687800
	82	3403	114164	546640	1739000
	83	3486	118946	562900	1791000
	84	3570	123870	579400	1843800
	85	3655	128938	596140	1897400
	86	3741	134152	613120	1951800
	87	3828	139514	630340	2007000
	88	3916	145026	647800	2063000
	89	4005	150690	665500	2119800
	90	4095	156508	683440	2177400
	91	4186	162482	701620	2235800
	92	4278	168614	720040	2295000
	93	4371	174906	738700	2355000
	94	4465	181360	757600	2415800
	95	4560	187978	776740	2477400
	96	4656	194762	796120	2539800
	97	4753	201714	815740	2603000
	98	4851	208836	835600	2667000
	99	4950	216130	855700	2731800
	100	5050	223598	876040	2797400
	101	5151	231242	896620	2863800
	102	5253	239064	917440	2931000
	103	5356	247066	938500	2999000
	104	5460	255250	959800	3067800
	105	5565	263618	981340	3137400
	106	5671	272172	1003120	3207800
	107	5778	280914	1025140	3279000
	108	5886	289846	1047400	3351000
	109	5995	298970	1069900	3423800
	110	6105	308288	1092640	3497400
	111	6216	317802	1115620	3571800
	112	6328	327514	1138840	3647000
	113	6441	337426	1162300	3723000
	114	6555	347540	1185900	3799800
	115	6670	357858	1209640	3877400
	116	6786	368382	1233520	3955800
	117	6903	379114	1257540	4035000
	118	7021	390056	1281700	4115000
	119	7140	401210	1306000	4195800
	120	7260	412578	1330440	4277400
	121	7381	424162	1355020	4359800
	122	7503	435964	1379740	4443000
	123	7626	447986	1404600	4527000
	124	7750	460230	1429600	4611800
	125	7875	472698	1454740	4697400
	126	8001	485392	1480020	4783800
	127	8128	498314	1505440	4871000
	128	8256	511466	1531000	4959000
	129	8385	524850	1556700	5047800
	130	8515	538468	1582540	5137400
	131	8646	552312	1608520	5227800
	132	8778	566394	1634640	5319000
	133	8911	580716	1660900	5411000
	134	9045	595280	1687300	5503800
	135	9180	610088	1713840	5597400
	136	9316	625142	1740520	5691800
	137	9453	640454	1767340	5787000
	138	9591	656026	1794300	5883000
	139	9730	671860	1821400	5979800
	140	9870	687958	1848640	6077400
	141	10011	704322	1876020	6175800
	142	10153	720954	1903540	6275000
	143	10296	737856	1931200	6375000
	144	10440	755030	1959000	6475800
	145	10585	772478	1986940	6577400
	146	10731	790202	2015020	6679800
	147	10878	808204	2043240	6783000
	148	11026	826486	2071600	6887000
	149	11175	845050	2100100	6991800
	150	11325	863898	2128740	7097400
	151	11476	883032	2157520	7203800
	152	11628	902454	2186440	7311000
	153	11781	922166	2215500	7419000
	154	11935	942170	2244700	7527800
	155	12090	962468	2274040	7637400
	156	12246	983062	2303520	7747800
	157	12403	1003954	2333140	7859000
	158	12561	1025146	2362900	7971000
	159	12720	1046640	2392800	8083800
	160	12880	1068438	2422840	8197400
	161	13041	1090542	2453020	8311800
	162	13203	1112954	2483340	8427000
	163	13366	1135676	2513800	8543000
	164	13530	1158710	2544400	8659800
	165	13695	1182058	2575140	8777400
	166	13861	1205722	2606020	8895800
	167	14028	1229704	2637040	9015000
	168	14196	1253996	2668200	9135000
	169	14365	1278600	2699500	9255800
	170	14535	1303518	2730940	9377400
	171	14706	1328752	2762520	9499800
	172	14878	1354304	2794240	9623000
	173	15051	1380176	2826100	9747000
	174	15225	1406370	2858100	9871800
	175	15400	1432888	2890240	9997400
	176	15576	1459732	2922520	10123800
	177	15753	1486904	2954940	10251000
	178	15931	1514406	2987500	10379000
	179	16110	1542240	3020200	10507800
	180	16290	1570418	3053040	10637400
	181	16471	1598942	3086020	10767800

3.4.2 Problem

Determine the number of paths from A to B, traveling downward and to the right.



4 Probability

4.1 Definition

The probability of an event is a number that indicates how likely the event is to occur.
Mathematical Definition:

$$P(A) = \frac{\text{Number of times A happens}}{\text{Sample Space}}$$

4.1.1 Discrete

- Example 1: A pair of dice are rolled, what is the probability of getting a sum of 7?

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

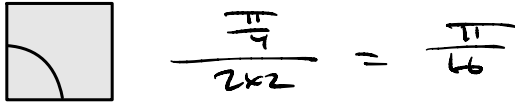
$$\frac{6}{36} = 1/6$$

- Example 2: 6 coins are tossed, what is the probability of getting a 2 head?

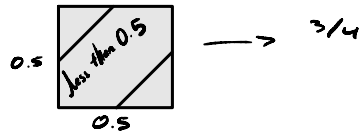
$$= \frac{\binom{6}{2}}{2^6}$$

4.1.2 Continuous

- Example 1: Point A is chosen randomly in $[0, 2] \times [0, 2]$, what's the probability that OA is less than 1.?



- Example 2: 2 points are chosen randomly in $[0, 1]$, what's the probability that their distance is less than 0.5.



- Example 3: Point A is chosen randomly in $[0, 1]$, what's the probability that A is 0.25.?

$$\therefore \text{Point} = \text{length of } 0$$

$$\therefore P = \frac{0}{1} = \underline{\underline{0}}$$

4.2 Some Formulae about Probability

- $P(E \cap F) = P(E)P(F)$ if E and F are independent events.
- $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
- $P(E|F) = \frac{P(E \cap F)}{P(F)}$.

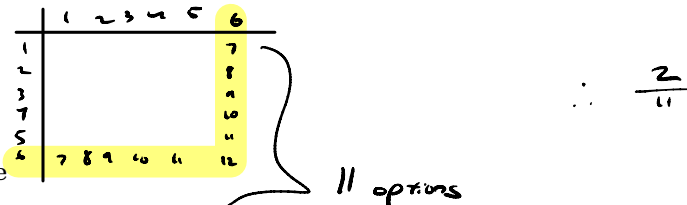
4.2.1 Example

What's the probability of tossing 3 heads in a row?

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

4.2.2 Example

A pair of dice are rolled, what is the probability of getting a sum of 7 given that one of them is 6?



4.2.3 Example

An integer is randomly chosen in $[1, 100]$, what is the probability that it is a multiple of 2 or 5?

$$\begin{aligned} \text{mtp. } 2 &= \frac{1}{2} = 50 \\ \text{mtp. } 5 &= \frac{1}{5} = 20 \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{mtp. } 2 &= \frac{1}{2} = 50 \\ \text{mtp. } 5 &= \frac{1}{5} = 20 \end{aligned}} \right\} 70$$

$$\therefore 60\%$$

$$70 - 10 = 60$$

both mtp. 5 and 2 10

double count values

4.3 Expected Value

$$E = \sum \text{outcome} \times P(\text{outcome})$$

4.3.1 Example

A local club plans to invest \$10000 to host a baseball game. They expect to sell tickets worth \$15000. But if it rains on the day of game, they won't sell any tickets and the club will lose all the money invested. If the weather forecast for the day of game is 20% possibility of rain, is this a good investment?

$$EV = 15000 \times 80\% + 0 \times 20\% \\ = 12000$$

$\therefore 12000 > 10000, \therefore$ good investment

4.3.2 Example

If Adam rolls a die repeatedly until he obtains a 6, what is the anticipated number of times he will need to roll the die?

$$E = \frac{1}{6} + 2 \times \frac{5}{6} \times \frac{1}{6} + 3 \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \dots$$

$$E = \frac{q}{p} + 1 = \frac{5/6}{1/6} + 1 = 6$$

\therefore He will need to roll the die 6 times to obtain a 6 on average.