Algebra 2 - Functional Equations

TSS Math Club

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1 Introduction:

1.1 Basic Types of Functions:

- Linear
- Quadratic
- Square Root
- Reciprocal

1.2 Equations:

Statement that shows two (or more) mathematical expressions are equal.
For example, solve for x: $2x = 10 - \frac{x}{4}$

In a functional equation, the unknown is not a value, it is a function. To solve a functional equation, find the relationship between the input x and the output $f(x)$

1.3 Functional Equations:

There are usually two conditions given:

- Equation
- Domain and Range

Any solution given must satisfy those two conditions.

2 Basic Examples

2.1 Solving Functional Equations

- Substitution
- Induction

2.2 Example

$$f(x+3) = x^2 + 5x$$

Determine $f(x)$

let $u = x + 3$

$x = u - 3$

$f(u) = (u-3)^2 + 5(u-3)$

$= u^2 - 6u + 9 + 5u - 15$

$= u^2 - u - 6$

$\therefore f(x) = x^2 - x - 6$

2.3 Example

$$f(\frac{2x-1}{x-3}) = x^2$$

Determine $f(x)$

$u = \frac{2x-1}{x-3}$

$(x-3)u = 2x-1$

$xu - 3u = 2x - 1$

$x(u-2) - 3u = -1$

$x = \frac{3u-1}{u-2}$

$f(u) = \left(\frac{3u-1}{u-2}\right)^2$

$f(x) = \left(\frac{3x-1}{x-2}\right)^2$

Extra:

$f(x) = 2f(1-x) + x$

Sub x by $1-x$:

$f(1-x) = 2f(x) + 1 - x$

Sub:

$f(x) = 2(2f(x) + 1 - x) + x$

$f(x) = 4f(x) + 2 - x$

$\frac{(x-2)}{3} = \frac{\cancel{3}f(x)}{\cancel{3}}$

3 Problems

3.1 Cauchy's Functional Equation

Find all functions $f : \mathbf{Q} \to \mathbf{Q}$ such that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbf{Q}$

$x=y=0$

$f(0+0) = f(0) + f(0)$

$0 = f(0)$

$x=y=1$

$f(2x) = 2f(x)$

$f(2) = f(1)+f(1)$, let $f(1)=k$

$= 2k$

$f(3) = f(1+2)$

$= f(1) + f(2)$

$= k+2k$

$= 3k$

Induction:

Base case: $f(1)=k$

Assume $f(n) = nk$

$f(n+1) = (n+1)k$

$f(n+1)$, $x=n, y=1$

$= f(n) + f(1)$

$= nk + k$

$= k(n+1)$

by induction, $f(n)=nk$ for $n \in \mathbb{N}$

let $y=-x$

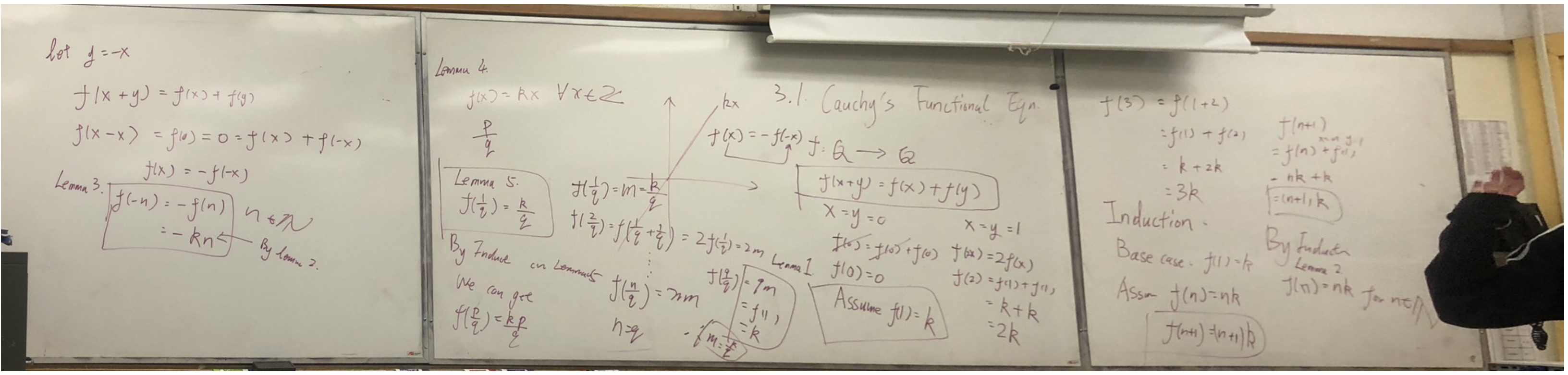
$f(x+y) = f(x)+f(y)$

$f(x-x) = f(0) = 0$

$= f(x)+f(-x)$

$f(-n) = -f(n)$, $n \in \mathbb{N}$

$= -kn$



3.2 Problem

$$f(x-y) = f(x) + f(y) - 2xy$$

$x=y=0$

$f(0) = 2f(0) - 0$

$f(0) = 0$

Sub $x=y$

$f(0) = f(x)+f(x) - 2x^2$

$2f(x) = 2x^2$

$f(x) = x^2$

3.3 2020 CSMC A6

Suppose that $f(x)$ is a function defined for every real number x with $0 \le x \le 1$ with the properties that:

① • $f(1-x) = 1 - f(x)$ for all real numbers x with $0 \le x \le 1$,

② • $f(\frac{1}{3}x) = \frac{1}{2}f(x)$ for all real numbers x with $0 \le x \le 1$, and

③ • $f(a) \le f(b)$ for all real numbers $0 \le a \le b \le 1$.

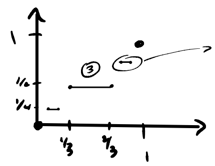
Cannot decrease in value between 2 points.

What is the value of $f(\frac{6}{7})$?

$x = 0$

② $f(0) = 0$

① $f(1) = 1$



$f(\frac{6}{7}) = 3/4$

② let $x = 1$

$f(\frac{1}{3}) = \frac{1}{2}f(1)$

$= \frac{1}{2}$

① $f(\frac{2}{3}) = 1 - f(\frac{1}{3})$

$= 1 - 1/2$

$= 1/2$

② let $x = \frac{1}{3}$

$f(\frac{1}{9}) = 1/2 f(\frac{1}{3})$

$= 1/4$

$f(\frac{8}{9}) = 1 - f(\frac{1}{3})$

$= 1 - 1/4$

$= 3/4$

3.4 2021 CSMC B3

A pair of functions $f(x)$ and $g(x)$ is called a *Payneful pair* if:

(i) $f(x)$ is a real number for all real numbers x,

(ii) $g(x)$ is a real number for all real numbers x,

(iii) $f(x+y) = f(x)g(y) + g(x)f(y)$ for all real numbers x and y,

(iv) $g(x+y) = g(x)g(y) - f(x)f(y)$ for all real numbers x and y, and

(v) $f(a) \neq 0$ for some real number a.

For every *Payneful pair* of functions $f(x)$ and $g(x)$:

(a) Determine the values of $f(0)$ and $g(0)$.

(b) If $h(x) = (f(x))^2 + (g(x))^2$, for all real numbers x, determine the value of $h(5)h(-5)$.

(c) If $-10 \leq f(x) \leq 10$ and $-10 \leq g(x) \leq 10$, for all real numbers x, determine the value of $h(2021)$.

$x = y = 0$

$f(0) = 2f(0)g(0)$

$f(0) = 0$ or not

$g(0) = g^2(0) - f^2(0)$

$x = y$

$f(2x) = 2f(x)g(x)$

$x = -y$

$f(0) = f(x)g(-x) + g(x)f(-x)$

$x = 0$

$f(y) = f(0)g(y) + g(0)f(y)$