Algebra 2 - Functional Equations

TSS Math Club

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1 Introduction:

1.1 Basic Types of Functions:

- Linear
- Quadratic
- Square Root
- Reciprocal

1.2 Equations:

Statement that shows two (or more) mathematical expressions are equal.
For example, solve for x: $2x = 10 - \frac{x}{4}$

In a functional equation, the unknown is not a value, it is a function. To solve a functional equation, find the relationship between the input x and the output $f(x)$

1.3 Functional Equations:

There are usually two conditions given:

- Equation
- Domain and Range

Any solution given must satisfy those two conditions.

2 Basic Examples

2.1 Solving Functional Equations

- Substitution
- Induction

2.2 Example

$$f(x+3) = x^2 + 5x$$

Determine $f(x)$

2.3 Example

$$f(\frac{2x-1}{x-3}) = x^2$$

Determine $f(x)$

3 Problems

3.1 Cauchy's Functional Equation

Find all functions $f : \mathbf{Q} \to \mathbf{Q}$ such that

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbf{Q}$

3.2 Problem

$$f(x - y) = f(x) + f(y) - 2xy$$

3.3 2020 CSMC A6

Suppose that $f(x)$ is a function defined for every real number x with $0 \leq x \leq 1$ with the properties that:

- $f(1-x) = 1 - f(x)$ for all real numbers x with $0 \leq x \leq 1$,
- $f(\frac{1}{3}x = \frac{1}{2}f(x)$ for all real numbers x with $0 \leq x \leq 1$, and
- $f(a) \leq f(b)$ for all real numbers $0 \leq a \leq b \leq 1$.

What is the value of $f(\frac{6}{7})$?

3.4 2021 CSMC B3

A pair of functions $f(x)$ and $g(x)$ is called a *Payneful pair* if:

(i) $f(x)$ is a real number for all real numbers x,

(ii) $g(x)$ is a real number for all real numbers x,

(iii) $f(x+y) = f(x)g(y) + g(x)f(y)$ for all real numbers x and y,

(iv) $g(x+y) = g(x)g(y) - f(x)f(y)$ for all real numbers x and y, and

(v) $f(a) \neq 0$ for some real number a.

For every *Payneful pair* of functions $f(x)$ and $g(x)$:

(a) Determine the values of $f(0)$ and $g(0)$.

(b) If $h(x) = (f(x))^2 + (g(x))^2$, for all real numbers x, determine the value of $h(5)h(-5)$.

(c) If $-10 \leq f(x) \leq 10$ and $-10 \leq g(x) \leq 10$, for all real numbers x, determine the value of $h(2021)$.